MATH 2028 Honours Advanced Calculus II
2021-22 Term 1
Problem Set 3

due on Oct 4, 2021 (Monday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Gradescope on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: Throughout this problem set, we use R to denote a rectangle in $\mathbb{R}^n$. When we write $R = A \times B$, then we mean $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^k$ are rectangles with $n = m + k$.

Problems to hand in

1. Evaluate the following integrals:
   (a) $\int_R \frac{x}{x^2+y} \, dV$ where $R = [0,1] \times [1,3]$
   (b) $\int_0^1 \int_{x^2}^x \frac{x}{1+y^2} \, dydx$
   (c) $\int_0^1 \int_{\sqrt{y}}^1 e^{y/x} \, dxdy$
2. Find the volume of the region in $\mathbb{R}^3$ bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.
3. Let P and Q be bounded subsets of $\mathbb{R}^3$ whose boundaries have measure zero. For each $z_0 \in \mathbb{R}$, we define the following subsets of $\mathbb{R}^2$:
$$P_{z_0} := \{(x,y) \in \mathbb{R}^2 \mid (x,y,z_0) \in P\},$$
$$Q_{z_0} := \{(x,y) \in \mathbb{R}^2 \mid (x,y,z_0) \in Q\}.$$
   Suppose for EACH $z_0 \in \mathbb{R}$, the boundaries ∂P_{z_0} and ∂Q_{z_0} have measure zero in $\mathbb{R}^2$ and we have Area(P_{z_0}) =Area(Q_{z_0}). Prove that Vol(P) =Vol(Q).
4. Let $f : R = A \times B \to \mathbb{R}$ be a bounded integrable function. Suppose $g : A \to \mathbb{R}$ is a function satisfying
$$\underline{\int}_B f(x,y) \, dy \leq g(x) \leq \overline{\int}_B f(x,y) \, dy$$
   for all $x \in A$. Prove that g is integrable over A and $\int_A g(x) \, dx = \int_R f \, dV$.

Suggested Exercises

1. Evaluate the following integrals:
   (a) $\int_R \frac{y}{x} \, dV$ where $R = [1,3] \times [2,4]$
   (b) $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y \, dxdy$
   (c) $\int_0^4 \int_{\sqrt{y}}^2 \frac{1}{1+x^3} \, dxdy$

2. Find the volume of the region in $\mathbb{R}^3$ bounded below by the xy-plane, above by $z = y$, and on the sides by $y = 4 - x^2$.

3. Let $\Omega \subset \mathbb{R}^3$ be the portion of the cube $[0,1] \times [0,1] \times [0,1]$ lying above the plane $y + z = 1$ and below the plane $x + y + z = 2$. Evaluate the integral $\int_\Omega x \, dV$.

4. Let $f : \Omega \to \mathbb{R}$ be a C^2 function [1] on an open subset $\Omega \subset \mathbb{R}^2$. Use Fubini's Theorem to prove that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ everywhere in Ω.

5. Let $f : R = [a,b] \times [c,d] \to \mathbb{R}$ be a continuous function. Define another function $F : R \to \mathbb{R}$ such that
$$F(x,y) := \int_{[a,x]\times[c,y]} f \, dV.$$
Compute $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ in the interior of R.

6. Let $f : R = [a,b] \times [c,d] \to \mathbb{R}$ be a continuous function such that $\frac{\partial f}{\partial y}$ is continuous on R. Define $G : [c,d] \to \mathbb{R}$ such that
$$G(y) := \int_a^b f(x,y) \, dx.$$
   (a) Show that G is continuous on $[c,d]$.
   (b) Prove that G is differentiable on (c,d) and $G'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y) \, dx$.

7. Let $f(x) = \left(\int_0^x e^{-t^2} \, dt \right)^2$ and $g(x) = \int_0^1 (t^2+1)^{-1} e^{-x^2(t^2+1)} \, dt$.
   (a) Prove that $f(x) + g(x) = \frac{\pi}{4}$ for all $x \geq 0$.
   (b) Use (a) to show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

8. Let $f : R = [0,1] \times [0,1] \to \mathbb{R}$ be the function defined by
$$f(x,y) = \begin{cases} 1 & \text{if } y \in \mathbb{Q}, \\ 2x & \text{if } y \notin \mathbb{Q}. \end{cases}$$
   (a) Prove that f is NOT integrable on R.
   (b) Show that each iterated integral $\int_0^1 \int_0^1 f(x,y) \, dxdy$ and $\int_0^1 \overline{\int_0^1} f(x,y) \, dydx$ exist and compute their values.

Challenging Exercises

1. (a) Let $C \subset R = A \times B$ be a set of content zero in $\mathbb{R}^n$. Let A' be the set of all $x \in A$ such that the set $\{y \in B \mid (x,y) \in C\}$ does NOT have content zero in $\mathbb{R}^k$. Prove that A' is a set of measure zero in $\mathbb{R}^m$.
   (b) Let $C \subset R = [0,1] \times [0,1]$ be the set consisting of all $(x,y) \in R$ such that $x = \frac{p}{q} \in \mathbb{Q}$, where $p, q \in \mathbb{N}$ are coprime, and $y \in [0, \frac{1}{q}]$. Prove that C has content zero in $\mathbb{R}^2$ but the subset A' as defined in (a) does NOT have content zero in $\mathbb{R}$.

[1] Recall that a function f is C^k if all the partial derivatives up to order k exist and are continuous.